

COMPLETE INTERSECTIONS WITH S^1 -ACTION

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ABSTRACT. We give the diffeomorphism classification of complete intersections with S^1 -symmetry in dimension ≤ 6 . In particular, we show that a 6-dimensional complete intersection admits a smooth non-trivial S^1 -action if and only if it is diffeomorphic to the complex projective space or the quadric.

1. INTRODUCTION

A complete intersection $X_n(d_1, \dots, d_r) \subset \mathbb{C}P^{n+r}$ is a smooth $2n$ -dimensional manifold given by a transversal intersection of r non-singular hypersurfaces in complex projective space. The hypersurfaces are defined by homogeneous polynomials whose degrees are given by an unordered r -tuple d_1, \dots, d_r . In general, the induced complex structure of $X_n(d_1, \dots, d_r)$ depends on the choice of the polynomials. However, as noted by Thom, the diffeomorphism type of a complete intersection only depends on n and the multi-degree (d_1, \dots, d_r) .

Complete intersections play a prominent rôle in algebraic geometry. In topology their classification up to diffeomorphism, homeomorphism or homotopy has been an active research area for many decades (see for example [17, 18], [16, Section 8]). In this paper we consider the

Question 1.1. *Which complete intersections admit a smooth non-trivial S^1 -action?*

The two dimensional complete intersections with S^1 -symmetry are diffeomorphic to the sphere or the torus and are given by $X_1(1) \cong X_1(2) \cong S^2$ and $X_1(3) \cong X_1(2, 2) \cong S^1 \times S^1$. In dimension four Seiberg-Witten theory can be applied to show that a complete intersection with S^1 -symmetry is diffeomorphic to $X_2(1)$, $X_2(2)$, $X_2(3)$ or $X_2(2, 2)$, i.e. to a complete intersection with positive first Chern class (see the next section for details).

The main purpose of this note is to answer Question 1.1 in dimension 6.

Theorem 1.2. *A 6-dimensional complete intersection $X_3(d_1, \dots, d_r)$ admits a smooth non-trivial S^1 -action if and only if $X_3(d_1, \dots, d_r)$ is diffeomorphic to the complex projective space $X_3(1)$ or the quadric $X_3(2)$.*

In particular, some 6-dimensional complete intersections, like the cubic $X_3(3)$ or the quartic $X_3(4)$, have positive first Chern class but do not admit a smooth non-trivial S^1 -action.

For Hamiltonian circle actions Theorem 1.2 maybe deduced from recent work [21] of Tolman on the classification of Hamiltonian circle actions on symplectic 6-manifolds with $b_2 = 1$.

Theorem 1.2 follows from a more general statement about 6-manifolds (see Theorem 2.4) which we prove using methods from equivariant cohomology and equivariant index theory.

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In higher dimensions a classification of complete intersections with S^1 -symmetry is not known (at least to the authors). Examples with S^1 -symmetry, which come into mind, are the complex projective space and the quadric, which are diffeomorphic to $SU(n+1)/S(U(n) \times U(1))$ and $SO(n+2)/SO(n) \times SO(2)$, respectively.

An important obstruction to the existence of smooth S^1 -actions originates from spin geometry. By a theorem of Atiyah and Hirzebruch [3] the index of the Dirac operator, the \hat{A} -genus, vanishes on spin complete intersections with smooth non-trivial S^1 -action.

In dimension $2n = 4k$ the \hat{A} -genus of a spin complete intersection $X_n(d_1, \dots, d_r)$ vanishes if and only if $n + r + 1 - \sum_{j=1}^r d_j > 0$, i.e. if the first Chern class is positive. This was first shown by Brooks [7] who gave an explicit formula for the \hat{A} -genus in terms of n and the multi-degree (d_1, \dots, d_r) . In particular, the number of diffeomorphism types of complete intersections with smooth non-trivial S^1 -action is finite if one restricts to spin complete intersections and to a fixed dimension $4k$. Without these restrictions diffeomorphism finiteness is only known in special cases (see for example [9, Th. 5.1]).

Question 1.1 has been studied for S^1 -actions preserving some additional structure. For example, it is known [14, p. 82-90] that the automorphism group of the natural complex structure on a complete intersection $X_n(d_1, \dots, d_r)$ is finite if the first Chern class is negative, i.e. if $n + r + 1 < \sum_{j=1}^r d_j$. Therefore there is no non-trivial circle action on these complete intersections which preserves the complex structure.

The paper is structured as follows. In the next section we explain the aforesaid classification of 4-dimensional complete intersections with S^1 -symmetry and derive Theorem 1.2 from a more general theorem (see Theorem 2.4) about certain 6-manifolds. Section 3 contains some preliminary facts about 6-manifolds. The proof of Theorem 2.4 consists of a case by case study of the possible S^1 -fixed point configurations which occupies the remaining three sections. In the appendix we collect formulas from equivariant cohomology and equivariant index theory which are used in the proof.

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2. S^1 -ACTIONS ON LOW DIMENSIONAL COMPLETE INTERSECTIONS

In this section we state a theorem about certain 6-manifolds and apply this theorem to obtain the classification of 6-dimensional complete intersections with S^1 -symmetry mentioned in the introduction. We also discuss Question 1.1 in more detail for complete intersections in lower dimensions.

Let $M := X_n(d_1, \dots, d_r)$ be a complete intersection given by a transversal intersection of r non-singular hypersurfaces in $\mathbb{C}P^{n+r}$ of degree d_1, \dots, d_r . Since the diffeomorphism type of M only depends on n and the multi-degree (d_1, \dots, d_r) , we may always assume $d_j \geq 2$ if $r \geq 2$. In fact, up to diffeomorphism, intersection with hypersurfaces of degree one amounts to cutting down the dimension of the ambient complex projective space.

Let γ denote the restriction of the dual Hopf bundle over $\mathbb{C}P^{n+r}$ to M and let $x := c_1(\gamma) \in H^2(M; \mathbb{Z})$. For later reference we collect some properties of M which follow from the Lefschetz hyperplane theorem, Poincaré duality and properties of characteristic classes.

Proposition 2.1. (1) M is simply connected for $n > 1$.

- (2) $H^*(M; \mathbb{Z})$ is torsion-free and the homology of M and $\mathbb{C}P^n$ are equal outside the middle dimension.
- (3) $[x^n]_M = \prod_j d_j$, where $[\]_M$ denotes evaluation on the fundamental cycle.
- (4) The total Chern class of M is given by

$$c(M) = (1+x)^{n+r+1} \cdot \prod_{j=1}^r (1+d_j \cdot x)^{-1}.$$

In particular, $c_1(M) = (n+r+1 - \sum_j d_j) \cdot x$.

- (5) The total Pontrjagin class of M is given by

$$p(M) = (1+x^2)^{n+r+1} \cdot \prod_{j=1}^r (1+d_j^2 \cdot x^2)^{-1}.$$

In particular, $p_1(M) = (n+r+1 - \sum_j d_j^2) \cdot x^2$.

- (6) The Euler characteristic of M , $\chi(M)$, is equal to $[c_n(M)]_M$. For $n=1$, $\chi(M) = d_1 \cdot \dots \cdot d_r \cdot (2 - \sum_{j=1}^r (d_j - 1))$. For $n=3$, $\chi(M) < 0$ except for $M = X_3(1) = \mathbb{C}P^3$ and $M = X_3(2) = SO(5)/(SO(3) \times SO(2))$ which have Euler characteristic equal to 4.

For a proof of these properties see for example [12]. The inequality for the Euler characteristic stated in (6) may be deduced from [12, formula (5), p. 127], see also [8].

Before we consider the classification of 6-dimensional complete intersections with S^1 -symmetry we give more details for complete intersections in lower dimensions.

By the classical Lefschetz fixed point formula the Euler characteristic of a manifold with S^1 -action is equal to the Euler characteristic of the S^1 -fixed point manifold. Combining this with the classification of surfaces it follows that the only orientable 2-manifolds with S^1 -symmetry are the sphere S^2 and the torus $S^1 \times S^1$. Applying the formula for the Euler characteristic given in Proposition 2.1 (6) one finds that among 2-dimensional complete intersections only $X_1(1) \cong X_1(2) \cong S^2$ and $X_1(3) \cong X_1(2, 2) \cong S^1 \times S^1$ admit a smooth non-trivial S^1 -action.

In dimension four Seiberg-Witten theory leads to the following classification, probably well-known to the experts.

Theorem 2.2. *A 4-dimensional complete intersection $X_2(d_1, \dots, d_r)$ admits a smooth non-trivial S^1 -action if and only if $X_2(d_1, \dots, d_r)$ is diffeomorphic to a complex projective plane $X_2(1)$, a quadric $X_2(2)$, a cubic $X_2(3)$ or an intersection of two quadrics $X_2(2, 2)$.*

Note that these are precisely the 4-dimensional complete intersections with positive first Chern class. Since we couldn't find a proof for Theorem 2.2 in the literature, we give the argument here.

Proof. We first explain why $X_2(1)$, $X_2(2)$, $X_2(3)$ and $X_2(2, 2)$ admit a smooth non-trivial S^1 -action. For the complex projective space $X_2(1)$ and the quadric $X_2(2)$, which are homogeneous, this is obvious. One knows that $X_2(3)$ (resp. $X_2(2, 2)$) is obtained by blowing up $\mathbb{C}P^2$ at 6 (resp. 5) points in general position (cf. [17, page 653], [19], [15, Section 3.5]). Hence, $X_2(3) \cong \mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$ and $X_2(2, 2) \cong \mathbb{C}P^2 \# 5\overline{\mathbb{C}P^2}$. Since any connected sum of $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P^2}$'s admits a smooth non-trivial S^1 -action, the same holds for $X_2(3)$ and $X_2(2, 2)$.

To show that no other complete intersection admits a smooth non-trivial S^1 -action we combine certain facts about Seiberg-Witten invariants. For any 4-dimensional complete intersection M different from $X_2(1)$, $X_2(2)$, $X_2(3)$ and $X_2(2, 2)$ one knows that $b_2^+(M)$ (the dimension of the maximal subspace of $H^2(M)$ on which

the intersection form is positive definite) is greater than one (cf. [17, page 650]). Since M is Kähler with $b_2^+(M) > 1$ the Seiberg-Witten invariant for M with its preferred $Spin^c$ -structure is ± 1 by the pioneering work of Witten [23]. On the other hand, Baldridge showed in [4] that for any smooth closed 4-manifold with $b_2^+ > 1$ the Seiberg-Witten invariant vanishes if the manifold admits a circle action with fixed point. Since $\chi(M) > 0$ any S^1 -action on M must have a fixed point. Hence, M does not admit a smooth non-trivial S^1 -action. ■

Remark 2.3. (1) By Freedman's classification [10] of simply connected topological 4-manifolds and the classification of indefinite odd forms any non-spin complete intersection $X_2(d_1, \dots, d_r)$ is homeomorphic to a connected sum of $\mathbb{C}P^2$'s and $\overline{\mathbb{C}P^2}$'s and, hence, admits a non-trivial *continuous* S^1 -action. (2) The only spin complete intersection with non-trivial *smooth* S^1 -action is the quadric $X_2(2)$. This follows directly from the \hat{A} -vanishing theorem of Atiyah-Hirzebruch [3] and the formula for the first Pontrjagin class given in Proposition 2.1 (5). The \hat{A} -vanishing theorem does not apply in general to *continuous* S^1 -actions. However, it is known that the signature vanishes on a 4-dimensional spin complete intersection with *continuous* S^1 -action provided the involution in S^1 acts non-trivially and locally smoothly [20] or the number of orbit types near every S^1 -fixed point is at most four [13]. It follows that among spin complete intersections the quadric is the only one with such an action.

We now come to the classification of complete intersections with smooth S^1 -action in dimension 6. Theorem 1.2 is a consequence of the following theorem which will be proved in the remaining sections.

Theorem 2.4. *Let M be a closed smooth oriented 6-dimensional manifold with torsion-free homology, $b_1(M) = 0$, $H^2(M; \mathbb{Z}) = \langle x \rangle$, $p_1(M) = \rho \cdot x^2$ with $\rho \leq 0$, $x^3 \neq 0$ and $\chi(M) < 4$. Then M does not support a non-trivial smooth circle action.*

Proof of Theorem 1.2. The complex projective space $X_3(1)$ and the quadric $X_3(2)$ are homogeneous and, hence, admit a smooth non-trivial S^1 -action.

Now assume $M := X_3(d_1, \dots, d_r)$ is different from $X_3(1)$ and $X_3(2)$. Then, either $r = 1$ and $d_1 \geq 3$ or $r \geq 2$ and $d_j \geq 2 \forall j$. In view of Proposition 2.1 M satisfies all the conditions of Theorem 2.4 and, hence, M does not admit a smooth non-trivial S^1 -action. ■

Remark 2.5. The bound $\rho \leq 0$ in Theorem 2.4 cannot be weakened. This can be shown as follows.

There are linear S^1 -actions on S^3 which have one-dimensional fixed point sets. For a fixed point $x \in S^3$, the isotropy representation at x is completely arbitrary. By taking products of such actions we get an action of S^1 on $S^3 \times S^3$ with a two-dimensional fixed point set and arbitrary isotropy representations at the fixed points.

Moreover, we may restrict the action of $SO(5)$ on the complete intersection $X_3(2) = SO(5)/SO(3) \times SO(2)$ to a subgroup of $SO(5)$ isomorphic to S^1 such that $X_3(2)^{S^1}$ has a two-dimensional component.

Therefore we may form the S^1 -equivariant connected sum of $S^3 \times S^3$ and $X_3(2)$. This connected sum satisfies all the assumptions of Theorem 2.4 except that the first Pontrjagin class $p_1((S^3 \times S^3) \# X_3(2))$ is equal to x^2 . Therefore the bound on ρ cannot be weakened.

Remark 2.6. For a simply-connected 6-manifold satisfying the cohomological assumptions in Theorem 2.4 there are, by surgery theory, infinitely many pairwise

non-diffeomorphic smooth manifolds inside its homotopy type with nonpositive first Pontrjagin class (i.e. $p_1 = \rho \cdot x^2$ with $\rho \leq 0$). By Theorem 2.4 none of these admit a non-trivial smooth S^1 -action.

Remark 2.7. Using methods of this paper one can show that in the presence of a smooth non-trivial S^1 -action the only possible values for the first Pontrjagin class are x^2 and $4 \cdot x^2$. We hope to report on this and applications to the classification of 6-manifolds with S^1 -symmetry at some other occasion.

3. PRELIMINARIES FOR THE PROOF OF THEOREM 2.4

Let M be a closed smooth oriented 6-dimensional manifold with torsion-free homology, $b_1(M) = 0$ and $b_2(M) = 1$.

Let x be a generator of $H^2(M; \mathbb{Z})$. The manifolds we are interested in satisfy the following conditions:

- (1) The Euler characteristic of M satisfies $\chi(M) < 4$.
- (2) $p_1(M) = \rho \cdot x^2$ with $\rho \leq 0$.
- (3) $x^3 \neq 0$.

We fix the orientation of M such that $t := [x^3]_M > 0$. Here and throughout the paper $[\]_N$ denotes evaluation on the fundamental cycle of an oriented closed manifold N .

Note that $b_3(M)$ is even, since the intersection form is skew-symmetric. If M is simply connected, then by the structure result of Wall [22] M is diffeomorphic to the connected sum of a twisted complex projective space (with twist number t) and $b_3(M)/2$ copies of $S^3 \times S^3$.

We now assume that M admits a smooth effective S^1 -action. Since $[x^3]_M \neq 0$ it follows from the localization theorem that M^{S^1} is not empty (see appendix).

Next we consider the Leray-Serre spectral sequence for the Borel construction $M \hookrightarrow M_{S^1} \rightarrow BS^1$ with real coefficients. Here and in the following we use the shorthand notation $b_{\text{odd}} = \sum_{2k+1} b_{2k+1}$ and $b_{\text{ev}} = \sum_{2k} b_{2k}$.

Lemma 3.1. *The spectral sequence for the Borel construction $M \hookrightarrow M_{S^1} \rightarrow BS^1$ degenerates at the E_2 -level.*

Proof. Since $b_1(M) = 0$ all differentials d_r of the spectral sequence vanish on x . By multiplicativity of the differentials d_r vanishes on $E_r^{*,2*}$ for all r . Thus all differentials in the spectral sequence vanish except maybe on $E_r^{*,3}$. For dimensional reasons $E_2^{*,6} = E_\infty^{*,6}$. Hence it follows from the multiplicativity of the differentials that the spectral sequence degenerates at the E_2 -level. \blacksquare

In the following we identify $\mathbb{Z}/m\mathbb{Z}$ with the cyclic subgroup of order m in S^1 .

Proposition 3.2. $b_{\text{ev}}(M^{S^1}) = b_{\text{ev}}(M) = 4$. For a prime p and $l > 0$,

$$b_{\text{ev}}(M^{\mathbb{Z}/p^l\mathbb{Z}}) = \text{rk } H^{\text{ev}}(M^{\mathbb{Z}/p^l\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}) = \text{rk } H^{\text{ev}}(M; \mathbb{Z}/p\mathbb{Z}) = 4.$$

Proof. By Lemma 3.1 the spectral sequence for $M \hookrightarrow M_{S^1} \rightarrow BS^1$ degenerates at the E_2 -level which implies $b_{\text{ev}}(M^{S^1}) = b_{\text{ev}}(M) = 4$ (cf. [6, p. 374-375]).

Note that S^1 acts on $M^{\mathbb{Z}/p^l\mathbb{Z}}$ and $(M^{\mathbb{Z}/p^l\mathbb{Z}})^{S^1} = M^{S^1}$. We fix a large prime q such that the action of $\mathbb{Z}/q\mathbb{Z} \subset S^1$ satisfies $M^{\mathbb{Z}/q\mathbb{Z}} = M^{S^1}$, $b_i(M) = \text{rk } H^i(M; \mathbb{Z}/q\mathbb{Z})$, $b_i(M^{\mathbb{Z}/p^l\mathbb{Z}}) = \text{rk } H^i(M^{\mathbb{Z}/p^l\mathbb{Z}}; \mathbb{Z}/q\mathbb{Z})$ and $b_i(M^{S^1}) = \text{rk } H^i(M^{S^1}; \mathbb{Z}/q\mathbb{Z})$.

Recall from [6, p. 376-377] that for any smooth S^1 -manifold Z and any prime p one has the inequality $\text{rk } H^{\text{ev}}(Z^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}) \leq \text{rk } H^{\text{ev}}(Z; \mathbb{Z}/p\mathbb{Z})$. An easy induction argument shows that we have also the following inequality $\text{rk } H^{\text{ev}}(Z^{\mathbb{Z}/p^l\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}) \leq$

$\text{rk } H^{ev}(Z; \mathbb{Z}/p\mathbb{Z})$. Recall also that $b_i(X) \leq \text{rk } H^i(X; \mathbb{Z}/p\mathbb{Z})$ for any space X . As an application of these properties one obtains

$$\begin{aligned} b_{ev}(M^{S^1}) &= \text{rk } H^{ev}(M^{S^1}; \mathbb{Z}/q\mathbb{Z}) = \text{rk } H^{ev}((M^{\mathbb{Z}/p^l\mathbb{Z}})^{\mathbb{Z}/q\mathbb{Z}}; \mathbb{Z}/q\mathbb{Z}) \\ &\leq \text{rk } H^{ev}(M^{\mathbb{Z}/p^l\mathbb{Z}}; \mathbb{Z}/q\mathbb{Z}) = b_{ev}(M^{\mathbb{Z}/p^l\mathbb{Z}}) \\ &\leq \text{rk } H^{ev}(M^{\mathbb{Z}/p^l\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}) \leq \text{rk } H^{ev}(M; \mathbb{Z}/p\mathbb{Z}) \end{aligned}$$

Since the homology of M is torsion-free $\text{rk } H^{ev}(M; \mathbb{Z}/p\mathbb{Z}) = \text{rk } H^{ev}(M; \mathbb{Z}) = b_{ev}(M) = 4$. Since $b_{ev}(M^{S^1}) = 4$ all inequalities in the display formula above are in fact equalities. In particular,

$$b_{ev}(M^{\mathbb{Z}/p^l\mathbb{Z}}) = \text{rk } H^{ev}(M^{\mathbb{Z}/p^l\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}) = \text{rk } H^{ev}(M; \mathbb{Z}/p\mathbb{Z}) = 4$$

■

Corollary 3.3. *For any prime p and $l > 0$ the fixed point manifold $M^{\mathbb{Z}/p^l\mathbb{Z}}$ is orientable.*

Proof. If p is an odd prime this is trivial since the action of $\mathbb{Z}/p^l\mathbb{Z}$ on the normal bundle of the fixed point manifold induces a complex structure. If $p = 2$ the claim follows from Proposition 3.2. Let X be a connected component of $M^{\mathbb{Z}/2^l\mathbb{Z}}$. Since $\mathbb{Z}/2^l\mathbb{Z}$ acts orientation preserving X is even-dimensional, say $\dim X = 2k$. Since $b_{2i}(X) \leq \text{rk } H^{2i}(X; \mathbb{Z}/2\mathbb{Z})$ and $b_{ev}(M^{\mathbb{Z}/2^l\mathbb{Z}}) = \text{rk } H^{ev}(M^{\mathbb{Z}/2^l\mathbb{Z}}; \mathbb{Z}/2\mathbb{Z})$ by Proposition 3.2 we see that $b_{2k}(X) = \text{rk } H^{2k}(X; \mathbb{Z}/2\mathbb{Z}) = 1$. Hence, X is orientable. ■

Remark 3.4. If $F \subset M^{\mathbb{Z}/n\mathbb{Z}}$ is a 4-dimensional component then the corollary above can be applied for a prime p dividing n to see that F is orientable.

Next we recall the classical results for the Euler characteristic and signature. By the Lefschetz-fixed point formula for the Euler characteristic

$$(3.1) \quad \chi(M) = \chi(M^{S^1}) = \sum_{Z \subset M^{S^1}} \chi(Z),$$

where the sum runs over the connected components Z of M^{S^1} .

The S^1 -action induces a complex structure and an orientation on the normal bundle ν_Z of Z . For later reference we remark that with respect to this complex structure on the normal bundle the normal S^1 -weights at Z are all positive. We equip Z with the orientation which is compatible with the orientations of ν_Z and M .

It follows from the rigidity of the equivariant signature (see (A.10)) that

$$\text{sign}(M) = \text{sign}(M^{S^1}) = \sum_{Z \subset M^{S^1}} \text{sign}(Z).$$

Note that if we replace the S^1 -action by the inverse action (by composing the action with the isomorphism $S^1 \rightarrow S^1, \lambda \mapsto \lambda^{-1}$), the orientation of Z will change if and only if Z has codimension $\equiv 2 \pmod{4}$, i.e. if the dimension of Z is 0 or 4. In order to simplify the discussion we will make the following

Convention 3.5. If the S^1 -action has at least one isolated fixed point, then we choose one of them, denoted by pt , and replace the action by the inverse action, if necessary, so that pt has positive orientation.

Next consider the complex line bundle L over M with $c_1(L) = x$. Since $b_1(M) = 0$ we may lift the action to L (cf. [11], Cor. 1.2). We first consider a fixed lift of the S^1 -action. Let Z be a connected component of M^{S^1} . At a point in Z the fibre of L is a complex one-dimensional S^1 -representation whose isomorphism type is constant on Z . We denote the weight of this representation by $a_Z \in \mathbb{Z}$. With this notation the equivariant first Chern class of L restricted to Z is equal to $x|_Z + a_Z \cdot z$.

Note that the lift is not unique. For any $l \in \mathbb{Z}$ we can choose a lift of the S^1 -action to L such that the S^1 -equivariant first Chern class at the connected components is given by $\{x|_Z + (a_Z + l) \cdot z \mid Z \subset M^{S^1}\}$.

The proof of Theorem 2.4 will be by a case by case study of the possible S^1 -fixed point configurations. Recall that M^{S^1} is not empty since $[x^3]_M = t \neq 0$. Since the Euler characteristic of M is < 4 the case of isolated S^1 -fixed points cannot occur (see Proposition 3.2 and equation (3.1)). Now any connected component of M^{S^1} is an oriented submanifold of even codimension. By Proposition 3.2 we are left with the following cases:

- M^{S^1} is the disjoint union of two connected 4-dimensional manifolds, $M^{S^1} = N_1 \cup N_2$, and $H^{ev}(N_i; \mathbb{R}) \cong H^{ev}(S^4; \mathbb{R})$.
- M^{S^1} is the disjoint union of a connected 4-dimensional manifold and a connected surface, $M^{S^1} = N \cup X$, and $H^{ev}(N; \mathbb{R}) \cong H^{ev}(S^4; \mathbb{R})$, $H^{ev}(X; \mathbb{R}) \cong H^{ev}(S^2; \mathbb{R})$.
- M^{S^1} is the disjoint union of a connected 4-dimensional manifold N , with $H^{ev}(N; \mathbb{R}) \cong H^{ev}(S^4; \mathbb{R})$, and two points pt and q .
- M^{S^1} is the disjoint union of a connected 4-dimensional manifold N , with $H^{ev}(N; \mathbb{R}) \cong H^{ev}(\mathbb{CP}^2; \mathbb{R})$, and a point pt .
- M^{S^1} is a connected 4-dimensional manifold N , with $b_2(N) = 2$.
- M^{S^1} is the disjoint union of two connected surfaces, $M^{S^1} = X \cup Y$.
- M^{S^1} is the disjoint union of a connected surface X and two points pt and q .

We will show in the following sections that none of these cases can occur.

4. 4-DIMENSIONAL FIXED POINT COMPONENTS

The following lemma will be used at several places in the proof.

Lemma 4.1. *Let $F \subset M$ be an oriented submanifold of codimension 2 and $\gamma \cdot x \in H^2(M; \mathbb{Z})$ its Poincaré-dual. Then:*

- (1) $p_1(F) = (\rho - \gamma^2) \cdot (x|_F)^2$.
- (2) *If $\text{sign}(F) = 0$, then $(x|_F)^2 = 0$, $\gamma = 0$ and the Euler class of the normal bundle of $F \hookrightarrow M$ vanishes.*

Proof. Consider the normal bundle ν of F in M equipped with the orientation compatible with the orientations of F and M . Then the Euler class of ν is equal to $\gamma \cdot (x|_F)$ (where $x|_F$ denotes the restriction of x to F) and

$$\rho \cdot (x|_F)^2 = p_1(M)|_F = p_1(F) + \gamma^2 \cdot (x|_F)^2$$

This shows the first statement.

For the second statement, note that $\text{sign}(F) = 0$ implies

$$(\rho - \gamma^2) \cdot (x|_F)^2 = p_1(F) = 0.$$

Also

$$[(x|_F)^2]_F = [(\gamma \cdot x) \cdot x^2]_M = t \cdot \gamma.$$

Since $\rho \leq 0$ and $t \neq 0$ these two identities imply $\gamma = 0$ and $(x|_F)^2 = 0$. ■

We now discuss the five cases involving a 4-dimensional connected S^1 -fixed point component.

Case 1: Assume that M^{S^1} is the disjoint union of two connected 4-dimensional manifolds, $M^{S^1} = N_1 \cup N_2$, and $H^{ev}(N_i; \mathbb{R}) \cong H^{ev}(S^4; \mathbb{R})$. Since $b_2(N_i) = 0$ the local datum for $[x^3]_M$ vanishes at N_i for any lift of the S^1 -action to L (see formula (A.3) in the appendix). This gives a contradiction since $[x^3]_M = t \neq 0$.

Case 2: Assume that M^{S^1} is the disjoint union of a connected 4-dimensional manifold and a connected surface, $M^{S^1} = N \cup X$, and $H^{ev}(N; \mathbb{R}) \cong H^{ev}(S^4; \mathbb{R})$, $H^{ev}(X; \mathbb{R}) \cong H^{ev}(S^2; \mathbb{R})$. We fix the lift of the S^1 -action to L for which $a_X + l = 0$. Consider the localization of $[x^3]_M$. The choice of lift guarantees that the local datum at X vanishes (see formula (A.2)). Since $b_2(N) = 0$ the local datum at N also vanishes by formula (A.3). This gives a contradiction since $[x^3]_M = t \neq 0$.

Case 3: Assume that M^{S^1} is the disjoint union of a connected 4-dimensional manifold N , with $H^{ev}(N; \mathbb{R}) \cong H^{ev}(S^4; \mathbb{R})$, and two points pt and q . By Convention 3.5 pt has positive orientation. Let $a_N + l$, $a_{pt} + l$ and $a_q + l$ denote the S^1 -weights of L at the fixed point components (here a_N, a_{pt}, a_q are fixed and $l \in \mathbb{Z}$ depends on the choice of lift). We compute $[x^3]_M$ locally. Since $b_2(N) = 0$ the local datum at N vanishes and we get (see appendix for formulas involved):

$$t = [x^3]_M = \epsilon_q \cdot \frac{(a_q + l)^3}{n_{q,1} \cdot n_{q,2} \cdot n_{q,3}} + \frac{(a_{pt} + l)^3}{n_{pt,1} \cdot n_{pt,2} \cdot n_{pt,3}}.$$

This is an identity of polynomials in l . By comparing coefficients one deduces $t = 0$ contradicting our assumption on t .

Case 4: Assume that M^{S^1} is the disjoint union of a connected 4-dimensional manifold N , with $H^{ev}(N; \mathbb{R}) \cong H^{ev}(\mathbb{CP}^2; \mathbb{R})$, and a point pt . By Convention 3.5 pt has positive orientation. Since $0 = \text{sign}(M) = \text{sign}(M^{S^1}) = \text{sign}(pt) + \text{sign}(N)$ the signature of N is -1 . Hence, $[x^2_N]_N \leq 0$ and $[p_1(N)]_N = -3$ by the signature theorem. From Lemma 4.1 (1) one now obtains the contradiction $-3 = (\rho - \gamma^2) \cdot [x^2_N]_N \geq 0$.

Case 5: Assume that M^{S^1} is a connected 4-dimensional manifold N with $b_2(N) = 2$. We fix the lift of the S^1 -action to L for which $a_N + l = 0$. By formula (A.3) the local datum for $[x^3]_M$ at N vanishes. This gives a contradiction since $[x^3]_M = t \neq 0$.

Hence, M does not support a smooth S^1 -action with a 4-dimensional fixed point component.

5. TWO 2-DIMENSIONAL FIXED POINT COMPONENTS

In this section we discuss the case that M^{S^1} is the disjoint union of two connected surfaces, $M^{S^1} = X \cup Y$. As before let L be the complex line bundle over M with $c_1(L) = x$. We fix a lift of the S^1 -action to L such that S^1 acts trivially on the fibres of L over Y (i.e. a_Y vanishes). Since any other lift differs by a global weight the S^1 -weights at the connected components X and Y for a general lift are of the form $a_X + l$ and l , respectively, where $l \in \mathbb{Z}$ depends on the choice of the lift.

Let $x_{Z,1}$ (resp. $y_{Z,i} + n_{Z,i} \cdot z$) denote the tangential root (resp. normal roots) at a component $Z \subset M^{S^1}$. For later use we note the

Lemma 5.1. $[x]_X \neq 0$ and $[x]_Y \neq 0$.

Proof. Assume $[x|_X]_X = 0$. Consider the localization formula (A.2) for $[x^3]_M$. By restricting to the constant terms and the terms of degree one (in l) one obtains

$$t = [x^3]_M = -\frac{a_X^3}{n_{X,1} \cdot n_{X,2}} \cdot \left(\frac{[y_{X,1}]_X}{n_{X,1}} + \frac{[y_{X,2}]_X}{n_{X,2}} \right) \text{ and}$$

$$0 = -3\frac{a_X^2}{n_{X,1} \cdot n_{X,2}} \cdot \left(\frac{[y_{X,1}]_X}{n_{X,1}} + \frac{[y_{X,2}]_X}{n_{X,2}} \right),$$

which contradicts $t \neq 0$. Hence, $[x|_X]_X \neq 0$. The argument for $[x|_Y]_Y \neq 0$ is analogous. \blacksquare

To prove the non-existence of an S^1 -action with $M^{S^1} = X \cup Y$ we first assume that the S^1 -action is semi-free around X and Y , i.e. we assume $n_{X,1} = n_{X,2} = n_{Y,1} = n_{Y,2} = 1$. In this case the S^1 -action on the normal bundles of X and Y coincides with complex multiplication by $S^1 \subset \mathbb{C}$ and the normal bundles of X and Y each split off a trivial complex line bundle on dimensional grounds. Hence, we may assume $y_{X,2} = y_{Y,2} = 0$, i.e. the normal weights at X (resp. Y) are $\{y_{X,1} + z, z\}$ (resp. $\{y_{Y,1} + z, z\}$).

We first compute $[x^3]_M$ locally. By formula (A.2)

$$t = [x^3]_M = -(a_X + l)^3 \cdot [y_{X,1}]_X + 3(a_X + l)^2 \cdot [x|_X]_X - l^3 \cdot [y_{Y,1}]_Y + 3 \cdot l^2 \cdot [x|_Y]_Y.$$

The left hand side is constant in l which gives the relations

$$[y_{Y,1}]_Y = -[y_{X,1}]_X, \quad a_X [y_{X,1}]_X = [x|_X]_X + [x|_Y]_Y, \quad a_X ([x|_X]_X - [x|_Y]_Y) = 0$$

$$\text{and } t = [x^3]_M = a_X^2 (-a_X \cdot [y_{X,1}]_X + 3 \cdot [x|_X]_X).$$

Next we compute $[p_1(M) \cdot x]_M$ locally. By formula (A.5)

$$\rho \cdot t = [p_1(M) \cdot x]_M = 2([x|_X]_X + [x|_Y]_Y).$$

This leads to

$$a_X \left(\frac{\rho \cdot t}{2} - 2 \cdot [x|_X]_X \right) = -a_X ([x|_X]_X - [x|_Y]_Y) = 0 \text{ and}$$

$$t = a_X^2 (-a_X \cdot [y_{X,1}]_X + 3 \cdot [x|_X]_X) = a_X^2 \left(-\frac{\rho \cdot t}{2} + 3[x|_X]_X \right).$$

Hence,

$$a_X \neq 0, \quad \rho \cdot t/2 = 2 \cdot [x|_X]_X \text{ and}$$

$$t = a_X^2 \left(-\frac{\rho \cdot t}{2} + \frac{3}{2} \cdot \frac{\rho \cdot t}{2} \right) = a_X^2 \left(\frac{1}{4} \cdot \rho \cdot t \right).$$

Since $t > 0$ and $\rho \leq 0$ the last equation gives a contradiction. Hence, the action on M cannot be semi-free around X and Y .

Next assume the action is not semi-free around X and Y . Then there exists a prim p and a 4-dimensional connected component $F \subset M^{\mathbb{Z}/p\mathbb{Z}}$ which contains one of the connected components of M^{S^1} , say X . Note that $b_2(F) \geq 1$ and $\text{rk } H^2(F; \mathbb{Z}/p\mathbb{Z}) \geq 1$ since $[x|_X]_X \neq 0$ by Lemma 5.1.

Lemma 5.2. $X \cup Y \subset F$

Proof. Suppose Y is contained in a connected component \tilde{F} of $M^{\mathbb{Z}/p\mathbb{Z}}$ which is different from F . Then $\text{rk } H^{ev}(M^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}) \geq \text{rk } H^{ev}(F; \mathbb{Z}/p\mathbb{Z}) + \text{rk } H^{ev}(\tilde{F}; \mathbb{Z}/p\mathbb{Z}) \geq 5$ which contradicts Proposition 3.2. Hence, $X \cup Y \subset F$. \blacksquare

By Corollary 3.3 F is orientable. We fix an orientation for F . Note that the signature of F vanishes since $\text{sign}(F) = \pm \text{sign}(X) \pm \text{sign}(Y) = 0$ (see (A.10)). By Lemma 4.1 the Euler class of the normal bundle of F vanishes. From the spectral sequence for $\pi : F_{S^1} \rightarrow BS^1$ we conclude that the equivariant Euler class is in the image of π^* . This implies that the S^1 -weights of the normal bundle at X and Y coincide. Since the normal bundle of $F \subset M$ restricted to X (resp. Y) is a

summand of the normal bundle of $X \subset M$ (resp. $Y \subset M$) we may assume that the normal roots of X (resp. Y) are given by $\{y_{X,1} + n_{X,1} \cdot z, n_{X,2} \cdot z\}$ (resp. $\{y_{Y,1} + n_{Y,1} \cdot z, n_{Y,2} \cdot z\}$), i.e. $y_{X,2} = y_{Y,2} = 0$ and $n_{X,2} = n_{Y,2}$.

To show that a smooth non-trivial S^1 -action does not exist we will first compute the S^1 -equivariant signature with the Lefschetz fixed point formula of Atiyah-Bott-Segal-Singer (see the appendix for details).

By the rigidity of the signature the S^1 -equivariant signature of M is zero. Using the Lefschetz fixed point formula we get (see formula (A.9)):

$$0 = \frac{1 + \lambda^{n_{X,2}}}{1 - \lambda^{n_{X,2}}} \cdot \frac{\lambda^{n_{X,1}}}{(1 - \lambda^{n_{X,1}})^2} \cdot [y_{X,1}]_X + \frac{1 + \lambda^{n_{X,2}}}{1 - \lambda^{n_{X,2}}} \cdot \frac{\lambda^{n_{Y,1}}}{(1 - \lambda^{n_{Y,1}})^2} \cdot [y_{Y,1}]_Y$$

By expanding the right hand side around $\lambda = 0$ one sees that either $[y_{X,1}]_X = 0 = [y_{Y,1}]_Y$ or $[y_{Y,1}]_Y = -[y_{X,1}]_X \neq 0$ and $n_{X,1} = n_{Y,1}$.

If $[y_{X,1}]_X = 0 = [y_{Y,1}]_Y$, then an inspection of the localization of $[x^3]_M$ using formula (A.2) gives the contradiction $0 \neq t = [x^3]_M = 0$.

If $[y_{Y,1}]_Y = -[y_{X,1}]_X \neq 0$, formula (A.2) gives the relations

$$[x|_X]_X = [x|_Y]_Y, \quad \frac{a_X}{n_{X,1}} \cdot [y_{X,1}]_X = 2 \cdot [x|_X]_X \text{ and } t = \frac{a_X^2}{n_{X,1} \cdot n_{X,2}} \cdot [x|_X]_X.$$

In particular, $[x|_X]_X$ is positive.

Assuming these relations the localization formula (A.5) for $[p_1(M) \cdot x]_M$ leads to

$$\rho \cdot t = 4 \cdot \frac{n_{X,1} \cdot [x|_X]_X}{n_{X,2}}.$$

Now $t > 0$, $\rho \leq 0$, $[x|_X]_X > 0$, $n_{X,i} > 0$ gives the desired contradiction.

Hence, M does not support a smooth S^1 -action with $M^{S^1} = X \cup Y$.

6. ONE 2-DIMENSIONAL FIXED POINT COMPONENT AND TWO ISOLATES FIXED POINTS

In this section we discuss the remaining case that M^{S^1} is the disjoint union of a connected surface X and two points pt and q . By our convention pt has positive orientation. Since the signature of M is equal to the sum of the signatures of the S^1 -fixed point components (see (A.10)) the orientation ϵ_q of the fixed point q is -1 .

Let $n_{X,1}, n_{X,2} > 0$ be the local weights at X , $n_{pt,1}, n_{pt,2}, n_{pt,3} > 0$ the local weights at pt and $n_{q,1}, n_{q,2}, n_{q,3} > 0$ the local weights at q . Since S^1 acts effectively on M , we have $\gcd(n_{X,1}, n_{X,2}) = \gcd(n_{pt,1}, n_{pt,2}, n_{pt,3}) = \gcd(n_{q,1}, n_{q,2}, n_{q,3}) = 1$.

We fix a lift of the S^1 -action on M into the line bundle L with $c_1(L) = x$ such that the weight a_X of the S^1 -representation on the fibers of L over X is zero.

Lemma 6.1. *The restriction $\iota^* : H^2(M; \mathbb{Z}) \rightarrow H^2(X; \mathbb{Z})$ is non-trivial.*

Proof. Assume that $\iota^*(x) = 0$ with $x \in H^2(M; \mathbb{Z})$ the preferred generator. Because the lift of the S^1 -action to L is not unique, we see with the localization formula for $[x^3]_M$ that $[x^3]_M$ is equal to a polynomial in the variable l . By comparing coefficients we get the following equations (see (A.1) and (A.2) in the appendix).

$$(6.1) \quad 0 = \frac{3a_{pt}}{n_{pt,1}n_{pt,2}n_{pt,3}} - \frac{3a_q}{n_{q,1}n_{q,2}n_{q,3}}$$

$$(6.2) \quad 0 = \frac{3a_{pt}^2}{n_{pt,1}n_{pt,2}n_{pt,3}} - \frac{3a_q^2}{n_{q,1}n_{q,2}n_{q,3}}$$

$$(6.3) \quad [x^3]_M = \frac{a_{pt}^3}{n_{pt,1}n_{pt,2}n_{pt,3}} - \frac{a_q^3}{n_{q,1}n_{q,2}n_{q,3}}$$

From equations (6.1) and (6.2), one sees that $a_{pt} = a_q$. Then it follows from equation (6.1) and (6.3) that $[x^3]_M = 0$. This is a contradiction to our assumption that $[x^3]_M > 0$. \blacksquare

Lemma 6.2. *Let F be the component of $M^{\mathbb{Z}/n_{X,1}\mathbb{Z}}$ which contains X . Then F contains both isolated fixed points. Moreover, F is orientable.*

Proof. If $n_{X,1} = 1$ the statement is trivially true. So let $n_{X,1} \geq 2$ and let p be a prime divisor of $n_{X,1}$. Because $\gcd(n_{X,1}, n_{X,2}) = 1$, F is the component of $M^{\mathbb{Z}/p\mathbb{Z}}$ which contains X . Moreover, F has dimension four. By Corollary 3.3 F is orientable.

By Proposition 3.2, we have $\text{rk } H^{ev}(M^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{Z}/p\mathbb{Z}) = 4$. By Lemma 6.1, we have $\text{rk } H^{ev}(F; \mathbb{Z}/p\mathbb{Z}) \geq 3$. Therefore, if $M^{\mathbb{Z}/p\mathbb{Z}}$ is disconnected, it is the union of F and a component F' with $b_{ev}(F') = \text{rk } H^{ev}(F'; \mathbb{Z}/p\mathbb{Z}) = 1$.

Assume that $M^{\mathbb{Z}/p\mathbb{Z}}$ is disconnected. It follows from the above discussion that $\text{sign}(F) = \pm 1$. This implies $[p_1(F)]_F = \pm 3$ and $[x^2_F]_F = \pm \alpha$ with $\alpha \geq 0$. By Lemma 4.1, we get the contradiction

$$3 = \pm[p_1(F)]_F = \pm(\rho - \gamma^2)[x^2_F]_F = (\rho - \gamma^2)\alpha \leq 0.$$

Therefore $F = M^{\mathbb{Z}/p\mathbb{Z}}$ is connected. \blacksquare

Lemma 6.3. *Let F be the connected fixed point component of the $\mathbb{Z}/n_{pt,1}\mathbb{Z}$ -action on M which contains pt . Then F also contains q and is orientable.*

Proof. If F is orientable then it contains X or q , because there is no orientable manifold which admits an S^1 -action with exactly one fixed point (see the appendix). In the first case $n_{pt,1}$ divides a local weight at X , say $n_{X,1}$. Then F contains also the component of $M^{\mathbb{Z}/n_{X,1}\mathbb{Z}}$ which contains X . Therefore it follows from Lemma 6.2 that F also contains q .

In the following we will show that F is always orientable. Let $k = \gcd(n_{pt,1}, n_{pt,2})$ and $k' = \gcd(n_{pt,1}, n_{pt,3})$. Note that

$$\gcd(k, n_{pt,3}) = \gcd(k', n_{pt,2}) = \gcd(n_{pt,1}, n_{pt,2}, n_{pt,3}) = 1.$$

Therefore there are $c_1, c_2, c_3 \in \mathbb{Z}$ such that

$$n_{pt,1} = c_1 k k' \quad n_{pt,2} = c_2 k \quad n_{pt,3} = c_3 k'.$$

If $n_{pt,1}$ is odd or $c_1 > 2$, then the normal bundle of F admits a complex structure. Therefore, F is orientable in this case.

If $c_1 = 2$, then there is some $l > 0$ such that F is a component of $M^{\mathbb{Z}/2^l\mathbb{Z}}$. Therefore it is orientable by Corollary 3.3.

Hence, we may assume that $c_1 = 1$, k is even and k' is odd. Then F is a component of $(M^{\mathbb{Z}/2\mathbb{Z}})^{\mathbb{Z}/k'\mathbb{Z}}$. Therefore it is orientable by Corollary 3.3. \blacksquare

Lemma 6.4. *The normal weights at pt and q are equal up to ordering.*

Proof. Up to ordering there are the following three cases.

- (1) $n_{pt,1} | n_{pt,2}$, $n_{pt,3} \nmid n_{pt,2}$, $\gcd(n_{pt,1}, n_{pt,3}) = 1$, $n_{pt,1} \neq 1$
- (2) $n_{pt,1} | n_{pt,2}$, $n_{pt,3} | n_{pt,2}$, $\gcd(n_{pt,1}, n_{pt,3}) = 1$, $n_{pt,1} \neq 1$
- (3) If $n_{pt,i} \neq 1$, then we have, for $j \neq i$, $n_{pt,i} \nmid n_{pt,j}$.

Before we consider these cases we prove the following two claims.

Claim 1: If $n_{pt,i_1} \nmid n_{pt,i_2}$ and $n_{pt,i_1} \nmid n_{pt,i_3}$, then there is exactly one $j \in \{1, 2, 3\}$ such that $n_{pt,i_1} | n_{q,j}$. Moreover, we have $n_{pt,i_1} = n_{q,j}$.

By Lemma 6.3, the component of $M^{\mathbb{Z}/n_{pt,i_1}\mathbb{Z}}$ which contains pt also contains q . Moreover, this component has dimension two. Therefore n_{pt,i_1} divides exactly one of the local weights at q , say $n_{q,j}$. Again by Lemma 6.3 the component of $M^{\mathbb{Z}/n_{q,j}\mathbb{Z}}$

which contains q also contains pt . Hence, $n_{q,j}$ divides one of the local weights at pt . It follows from the assumptions in the claim that this weight must be n_{pt,i_1} . Therefore $n_{pt,i_1} = n_{q,j}$ follows. This proves Claim 1.

Claim 2: If $n_{pt,i_1} | n_{pt,i_2}$ and $n_{pt,i_1} \nmid n_{pt,i_3}$, then there are exactly two $j_1, j_2 \in \{1, 2, 3\}$ such that $n_{pt,i_1} | n_{q,j_1}$ and $n_{pt,i_1} | n_{q,j_2}$. Moreover, we have $(n_{pt,i_1}, n_{pt,i_2}) = (n_{q,j_1}, n_{q,j_2})$ up to ordering.

By Lemma 6.3, the component of $M^{\mathbb{Z}/n_{pt,i_1}\mathbb{Z}}$ which contains pt also contains q . Moreover, this component has dimension four. Therefore n_{pt,i_1} divides exactly two of the local weights at q , say n_{q,j_1} and n_{q,j_2} .

At first assume $n_{pt,i_1} \neq n_{pt,i_2}$. Then, by Claim 1, applied to n_{pt,i_2} , we know that exactly one of these weights is equal to n_{pt,i_2} . Denote this weight by n_{q,j_2} . By Lemma 6.3 the component of $M^{\mathbb{Z}/n_{q,j_1}\mathbb{Z}}$ which contains q also contains pt . Therefore n_{q,j_1} divides n_{pt,i_1} or n_{pt,i_2} . In the second case this component has dimension four. Hence, n_{q,j_1} divides also n_{pt,i_1} . This implies $n_{pt,i_1} = n_{q,j_1}$.

Now assume that $n_{pt,i_1} = n_{pt,i_2}$. Then, by Lemma 6.3, the component of $M^{\mathbb{Z}/n_{q,j_1}\mathbb{Z}}$ which contains q also contains pt . Therefore n_{q,j_1} divides n_{pt,i_1} . Hence, $n_{q,j_1} = n_{pt,i_1}$ and by the same argument $n_{pt,i_1} = n_{q,j_2}$. This proves the second claim.

Now consider the three cases mentioned above. In the first case the statement of the lemma follows from Claim 1 applied to $n_{pt,i_1} = n_{pt,3}$ and Claim 2 applied to $n_{pt,i_1} = n_{pt,1}$.

In the situation of the second case at first assume that $n_{pt,3} \neq 1$. Then the statement of the lemma follows from Claim 2 applied to both $n_{pt,i_1} = n_{pt,1}$ and $n_{pt,i_1} = n_{pt,3}$. Now assume that $n_{pt,3} = 1$. Then it follows from Claim 2 applied to $n_{pt,i_1} = n_{pt,1}$ that there are two weights $n_{q,1}$ and $n_{q,2}$ such that $(n_{pt,1}, n_{pt,2}) = (n_{q,1}, n_{q,2})$ up to ordering. If we assume that $n_{q,3} \neq 1$, we get from Claim 1 or Claim 2 applied to $n_{pt,i_1} = n_{q,3}$ that there is a $n_{pt,j}$ which is equal to $n_{q,3}$. This leads to a contradiction because only two of the local weights at q are divisible by $n_{pt,1}$. Therefore the local weights at pt and q are the same up to ordering.

In the third case first apply Claim 1 to all $n_{pt,i} \neq 1$ to show that each of these weights is equal to exactly one local weight at q . As in the previous case it follows from an application of the Claims 1 and 2 to the local weights at q that $\#\{i; n_{pt,i} = 1\} = \#\{i; n_{q,i} = 1\}$. Therefore the lemma follows in this case. \blacksquare

Lemma 6.5. *The case $(n_{pt,1}, n_{pt,2}, n_{pt,3}) = (n_{q,1}, n_{q,2}, n_{q,3})$ does not occur.*

Proof. Assume that we are in this case. Since the lift of the S^1 -action into L is not unique we get from the localization formulas for $[x^3]_M$ and $[p_1(M) \cdot x]_M$ two polynomials in a variable l which are equal to $[x^3]_M$ and $[p_1(M) \cdot x]_M$, respectively. By comparing coefficients we get the following equations (see appendix).

$$(6.4) \quad 0 = \frac{3}{n_{X,1}n_{X,2}}[x]_X + \frac{3a_{pt}}{n_{pt,1}n_{pt,2}n_{pt,3}} - \frac{3a_q}{n_{pt,1}n_{pt,2}n_{pt,3}}$$

$$(6.5) \quad 0 = \frac{3a_{pt}^2}{n_{pt,1}n_{pt,2}n_{pt,3}} - \frac{3a_q^2}{n_{pt,1}n_{pt,2}n_{pt,3}}$$

$$(6.6) \quad [x^3]_M = \frac{a_{pt}^3}{n_{pt,1}n_{pt,2}n_{pt,3}} - \frac{a_q^3}{n_{pt,1}n_{pt,2}n_{pt,3}}$$

$$(6.7) \quad [p_1(M) \cdot x]_M = \frac{n_{X,1}^2 + n_{X,2}^2}{n_{X,1}n_{X,2}}[x]_X + (a_{pt} - a_q) \frac{n_{pt,1}^2 + n_{pt,2}^2 + n_{pt,3}^2}{n_{pt,1}n_{pt,2}n_{pt,3}}$$

Because of (6.5) we have $a_{pt} = \pm a_q$. Then (6.6) and $[x^3]_M > 0$ implies $a_{pt} = -a_q > 0$.

From (6.4) and (6.7) we get

$$[p_1(M) \cdot x]_M = -2a_{pt} \frac{n_{X,1}^2 + n_{X,2}^2}{n_{pt,1}n_{pt,2}n_{pt,3}} + 2a_{pt} \frac{n_{pt,1}^2 + n_{pt,2}^2 + n_{pt,3}^2}{n_{pt,1}n_{pt,2}n_{pt,3}}.$$

Now, using the divisibility properties implied by Lemma 6.2, it follows that the right hand side of this equation is always positive. This is a contradiction to our assumption. \blacksquare

By combining the above lemmas we see that there is no S^1 -action on M with a fixed point set consisting out of a two-dimensional component and two isolated fixed points.

APPENDIX A. LOCALIZATION FORMULAS FOR EQUIVARIANT COHOMOLOGY CLASSES AND EQUIVARIANT SIGNATURE

In the appendix we provide the local formulas for equivariant cohomology classes and equivariant signatures which are used in the proof.

Let Z be a connected component of the fixed point manifold M^{S^1} . We denote the tangential formal roots of Z by $x_{Z,j}$. Hence, the total Pontrjagin class of Z is given by $p(Z) = \prod_j (1 + x_{Z,j}^2)$.

Let ν_Z be the normal bundle of Z . Recall that, by our convention, ν_Z is oriented via the complex structure induced by the S^1 -action and that Z is oriented compatible to the orientation of M and ν_Z .

We denote the S^1 -equivariant Euler class of the normal bundle ν_Z by $e_{S^1}(\nu_Z)$. The normal bundle ν_Z decomposes as a direct sum of complex vector bundles corresponding to the S^1 -representations. Applying the splitting principle to these we can associate to ν_Z S^1 -equivariant roots $y_{Z,i} + n_{Z,i} \cdot z$, where the $y_{Z,i}$'s are non-equivariant formal roots of the corresponding bundle, the weights $n_{Z,i} \in \mathbb{Z}$ are positive by our convention, and z is a formal variable which one should think of as a fixed generator of the integral lattice of S^1 or a fixed generator of $H^2(BS^1; \mathbb{Z})$. With this notation the S^1 -equivariant Euler class of the normal bundle ν_Z is given by $e_{S^1}(\nu_Z) = \prod_i (y_{Z,i} + n_{Z,i} \cdot z)$.

A.1. Equivariant cohomology classes. We apply the localization formula in equivariant cohomology of Atiyah-Bott-Berline-Vergne [5, 1] to the classes x^3 and $p_1(M) \cdot x$. Fix a lift of the S^1 -action to the complex line bundle L with $c_1(L) = x$. Then at a connected component $Z \subset M^{S^1}$ the S^1 -equivariant first Chern class of L has the form $x|_Z + a_Z \cdot z$.

The lift is not unique. For any $l \in \mathbb{Z}$ we can choose a lift of the S^1 -action such that the S^1 -equivariant first Chern class at the connected components is given by $\{x|_Z + (a_Z + l) \cdot z \mid Z \subset M^{S^1}\}$.

The localization formula for x^3 with respect to such a lift gives

$$[x^3]_M = \sum_{Z \subset M^{S^1}} \mu(x^3, Z),$$

where the local datum $\mu(x^3, Z)$ at Z is given by

$$\mu(x^3, Z) = [(x|_Z + (a_Z + l) \cdot z)^3 \cdot e_{S^1}(\nu_Z)^{-1}]_Z.$$

Note that the sum $\sum_{Z \subset M^{S^1}} \mu(x^3, Z)$ is independent of the parameter l .

Depending on the dimension of Z the local datum $\mu(x^3, Z)$ for x^3 at Z takes the form:

If Z is a point, then

$$(A.1) \quad \mu(x^3, Z) = \epsilon_Z \cdot \frac{((a_Z + l) \cdot z)^3}{n_{Z,1} \cdot n_{Z,2} \cdot n_{Z,3} \cdot z^3} = \epsilon_Z \cdot \frac{(a_Z + l)^3}{n_{Z,1} \cdot n_{Z,2} \cdot n_{Z,3}},$$

where $\epsilon_Z \in \{\pm 1\}$ is $+1$ iff the point Z is positively oriented.

If Z is 2-dimensional, then

$$(A.2) \quad \begin{aligned} \mu(x^3, Z) &= \left[(x|_Z + (a_Z + l) \cdot z)^3 \cdot ((y_{Z,1} + n_{Z,1} \cdot z) \cdot (y_{Z,2} + n_{Z,2} \cdot z))^{-1} \right]_Z = \\ &= \frac{1}{n_{Z,1} \cdot n_{Z,2}} \cdot \left(-\frac{(a_Z + l)^3}{n_{Z,1}} \cdot [y_{Z,1}]_Z - \frac{(a_Z + l)^3}{n_{Z,2}} \cdot [y_{Z,2}]_Z + 3(a_Z + l)^2 \cdot [x|_Z]_Z \right). \end{aligned}$$

If Z is 4-dimensional, then

$$(A.3) \quad \begin{aligned} \mu(x^3, Z) &= \left[(x|_Z + (a_Z + l) \cdot z)^3 \cdot (y_{Z,1} + n_{Z,1} \cdot z)^{-1} \right]_Z = \\ &= \frac{3(a_Z + l)}{n_{Z,1}} \cdot [x|_Z]_Z - \frac{3(a_Z + l)^2}{n_{Z,1}^2} [x|_Z \cdot y_{Z,1}]_Z + \frac{(a_Z + l)^3}{n_{Z,1}^3} [y_{Z,1}^2]_Z \end{aligned}$$

Note that, if $a_Z + l = 0$, then the local data in (A.1), (A.2), (A.3) vanish. Note also that the local datum in (A.3) vanishes for any l if $b_2(Z) = 0$.

Next we provide formulas for $p_1(M) \cdot x$.

Depending on the dimension of Z the local datum $\mu(p_1(M) \cdot x, Z)$ for $p_1(M) \cdot x$ at Z takes the form:

If Z is a point, then the local datum is equal to $\epsilon_Z \cdot \frac{(a_Z + l) \cdot z \cdot (n_{Z,1}^2 + n_{Z,2}^2 + n_{Z,3}^2) \cdot z^2}{n_{Z,1} \cdot n_{Z,2} \cdot n_{Z,3} \cdot z^3}$, i.e.

$$(A.4) \quad \mu(p_1(M) \cdot x, Z) = \epsilon_Z \cdot \frac{(a_Z + l) \cdot (n_{Z,1}^2 + n_{Z,2}^2 + n_{Z,3}^2)}{n_{Z,1} \cdot n_{Z,2} \cdot n_{Z,3}},$$

where $\epsilon_Z \in \{\pm 1\}$ is $+1$ iff Z is positively oriented.

If Z is 2-dimensional, then the local datum is equal to

$$\begin{aligned} & \left[(x_{Z,1}^2 + (y_{Z,1} + n_{Z,1}z)^2 + (y_{Z,2} + n_{Z,2}z)^2) \cdot (x|_Z + (a_Z + l)z) \cdot \right. \\ & \quad \left. \cdot ((y_{Z,1} + n_{Z,1}z)(y_{Z,2} + n_{Z,2}z))^{-1} \right]_Z \end{aligned}$$

which gives

$$(A.5) \quad \begin{aligned} \mu(p_1(M) \cdot x, Z) &= -\frac{(a_Z + l)}{n_{Z,1} \cdot n_{Z,2}} (n_{Z,1}^2 + n_{Z,2}^2) \left(\frac{1}{n_{Z,1}} \cdot [y_{Z,1}]_Z + \frac{1}{n_{Z,2}} \cdot [y_{Z,2}]_Z \right) \\ &+ \frac{n_{Z,1}^2 + n_{Z,2}^2}{n_{Z,1} \cdot n_{Z,2}} \cdot [x|_Z]_Z + 2 \frac{(a_Z + l)}{n_{Z,1} \cdot n_{Z,2}} (n_{Z,1} \cdot [y_{Z,1}]_Z + n_{Z,2} \cdot [y_{Z,2}]_Z). \end{aligned}$$

If Z is 4-dimensional, then $\mu(p_1(M) \cdot x, Z)$ is equal to

$$(A.6) \quad \begin{aligned} & \left[(p_1(Z) + (y_{Z,1} + n_{Z,1} \cdot z)^2) \cdot (x|_Z + (a_Z + l) \cdot z) \cdot (y_{Z,1} + n_{Z,1} \cdot z)^{-1} \right]_Z \\ &= [x|_Z \cdot y_{Z,1}]_Z + \frac{a_Z + l}{n_{Z,1}} \cdot [p_1(Z)]_Z \end{aligned}$$

Note that, if $a_Z + l = 0$, then the local datum at Z vanishes if Z is a point. Note also that the local datum at Z vanishes for any l if the dimension of Z is 4 and $b_2(Z) = 0$.

A.2. Equivariant signatures. In this section we recall the Lefschetz fixed point formula for the equivariant signature and provide some formulas used in the paper.

Let M be an oriented closed manifold with smooth S^1 -action and let $\text{sign}_{S^1}(M)$ denote the S^1 -equivariant signature. A priori $\text{sign}_{S^1}(M)$ is an element of the representation ring $R(S^1)$ which we identify via the character with the ring of finite Laurent polynomials $\mathbb{Z}[\lambda, \lambda^{-1}]$.

By the Lefschetz fixed point formula of Atiyah-Bott-Segal-Singer (cf. [2]) the S^1 -equivariant signature can be computed locally at the S^1 -fixed point components. More precisely, for any topological generator $\lambda \in S^1$

$$(A.7) \quad \text{sign}_{S^1}(\lambda) = \sum_{Z \subset M^{S^1}} \mu_Z(\lambda),$$

where the local datum $\mu_Z(\lambda)$ at a connected component $Z \subset M^{S^1}$ is given by

$$\mu_Z(\lambda) = \left[\prod_j x_{Z,j} \cdot \frac{1 + e^{-x_{Z,j}}}{1 - e^{-x_{Z,j}}} \cdot \prod_i \frac{1 + \lambda^{-n_{Z,i}} \cdot e^{-y_{Z,i}}}{1 - \lambda^{-n_{Z,i}} \cdot e^{-y_{Z,i}}} \right]_Z.$$

For example, if M is 6-dimensional and Z is a point, then the local datum is given by

$$(A.8) \quad \mu_Z(\lambda) = \epsilon_Z \cdot \prod_{i=1}^3 \frac{1 + \lambda^{-n_{Z,i}}}{1 - \lambda^{-n_{Z,i}}}.$$

If Z is 2-dimensional, then the local datum is given by

$$(A.9) \quad \begin{aligned} \mu_Z(\lambda) &= \left[x_{Z,1} \cdot \frac{1 + e^{-x_{Z,1}}}{1 - e^{-x_{Z,1}}} \cdot \prod_{i=1}^2 \frac{1 + \lambda^{-n_{Z,i}} \cdot e^{-y_{Z,i}}}{1 - \lambda^{-n_{Z,i}} \cdot e^{-y_{Z,i}}} \right]_Z \\ &= 4 \cdot \left(\frac{1 + \lambda^{n_{Z,2}}}{1 - \lambda^{n_{Z,2}}} \cdot \frac{\lambda^{n_{Z,1}}}{(1 - \lambda^{n_{Z,1}})^2} \cdot [y_{Z,1}]_Z + \frac{1 + \lambda^{n_{Z,1}}}{1 - \lambda^{n_{Z,1}}} \cdot \frac{\lambda^{n_{Z,2}}}{(1 - \lambda^{n_{Z,2}})^2} \cdot [y_{Z,2}]_Z \right) \end{aligned}$$

By homotopy invariance the S^1 -equivariant signature is rigid, i.e. constant in λ . Hence, the sum $\sum_{Z \subset M^{S^1}} \mu_Z(\lambda)$ does not depend on λ (this can be shown also by comparing both sides of (A.7) and observing that poles of the left hand side can only occur in $0, \infty$, whereas a pole of the right hand side must be on the unit circle). In particular, in view of equation (A.8) S^1 cannot act on M with only one fixed point.

Recall from the beginning of the appendix that all $n_{Z,i}$ are positive. Taking the limit $\lambda \rightarrow \infty$ in the right hand side of (A.7) one sees that

$$(A.10) \quad \text{sign}(M) = \sum_{Z \subset M^{S^1}} \text{sign}(Z).$$

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